

## ON SPLITS OF COMPUTABLY ENUMERABLE SETS

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**ABSTRACT.** Our focus will be on the computably enumerable (c.e.) sets and trivial, non-trivial, Friedberg, and non-Friedberg splits of the c.e. sets. Every non-computable set has a non-trivial Friedberg split. Moreover, this theorem is uniform. V. Yu. Shavrukov recently answered the question which c.e. sets have a non-trivial non-Friedberg splitting and we provide a different proof of his result. We end by showing there is no uniform splitting of all c.e. sets such that all non-computable sets are non-trivially split and, in addition, all sets with a non-trivial non-Friedberg split are split accordingly.

## 1. TRIVIAL SPLITS

Given a c.e. set  $A$ , a *split* of  $A$  is a pair of c.e. sets  $A_0, A_1$  such that  $A_0 \sqcup A_1 = A$ ,  $\sqcup$  is disjoint union. If one of  $A_0$  or  $A_1$  is computable the splitting is *trivial*. If  $A_0$  is computable then  $A = A_0 \sqcup (\overline{A_0} \cap A)$ .

It is straightforward to see that any splitting of a computable set is trivial. Given a c.e. set  $A$ , letting  $A_0 = \emptyset$  and  $A_1 = A$ , provides a trivial splitting of  $A$ . We would like to avoid splits where one of the sets is finite. It is known that every infinite c.e. set  $A$  has an infinite computable subset  $R$ . This provides a trivial splitting of  $A$ ,  $A = R \sqcup (A \cap \overline{R})$ , into two infinite c.e. sets assuming  $A$  is not computable.

Given this, Myhill asked

**Question 1.1** (Myhill [9]). *Does every non-computable c.e. set have a non-trivial splitting?*

Myhill's Question was answered positively by Friedberg [5].

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Draft as of August 9, 2016. We want to thank V. Yu. Shavrukov for allowing us to include his result, Theorem 3.8. Without it, this paper would look very different. This research was started while Cholak participated in the Buenos Aires Semester in Computability, Complexity and Randomness, 2013. Thanks to Rachel Epstein, Greg Igusa, Nathan Pierson, Mike Stob, and the referees for comments and suggestions. My interest in Friedberg splits was sparked in 1989 by Rod Downey. I cannot forgive him.

## 2. FRIEDBERG SPLITS

Most of this section is known but we wanted to provide an explicit proof of Corollary 2.7. This corollary will be useful later. One of the focuses of this paper is splitting procedures that always produce a non-trivial split when possible.

At this point we will fix the standard uniform enumeration  $W_{e,s}$  of all c.e. sets with the convention that at stage  $s$ , there is at most one pair  $e, x$  where  $x$  enters  $W_e$  at stage  $s$ . Some details on how we can effectively achieve this enumeration can be found in Soare [12, Exercise I.3.11].

Every c.e. set has an index according to this fixed enumeration. For the sets that we construct we have to appeal to Kleene's Recursion Theorem to find this index. Moreover, by the standard trick of slowing down or pausing our construction, we can assume the enumerations of our fixed point  $W_e$  and our constructed set  $A$  are the same. Our construction, at times, will construct sets other than  $A$ . While we will focus on the constructed sets, the actual outcome of our constructions will be a uniform enumeration of all constructed sets. We will be using Kleene's Recursion Theorem with parameters to get a function from each constructed set to an index with the same enumeration for that set in the above enumeration.

By the Padding Lemma, we know that each c.e. set  $A$  has infinitely many indices. By Rice's Theorem, we know that for a given c.e. set  $A$  the set of indices (in this fixed enumeration) for  $A$  is not computable.

Also at this point we will fix the convention that  $A, B, W, X$  and  $Y$  always refer to c.e. sets with some fixed index in our given enumeration. Now we need the following.

**Definition 2.1.** A split  $A_0 \sqcup A_1 = A$  is a *Friedberg split* of  $A$  iff, for all  $W$ , if  $W - A$  is not c.e. then both  $W - A_0$  and  $W - A_1$  are not c.e. sets.

**Lemma 2.2.** *If  $A$  is not computable and  $A_0 \sqcup A_1$  is a Friedberg split then the split is not trivial.*

*Proof.*  $\mathbb{N} - A$  is not c.e. so  $\mathbb{N} - A_0$  and  $\mathbb{N} - A_1$  are not c.e. and hence  $A_0$  and  $A_1$  are not computable.  $\square$

**Definition 2.3.** For  $A = W_e$  and  $B = W_i$ ,

$$A \setminus B = \{x \mid \exists s [x \in (W_{e,s} - W_{i,s})]\}$$

and  $A \searrow B = A \setminus B \cap B$ . (This is with respect to our given enumeration and hence this definition depends on our chosen enumeration.)

By the above definition,  $A \setminus B$  is a c.e. set.  $A \setminus B$  is the set of balls that enter  $A$  before they enter  $B$ . If  $x \in A \setminus B$  then  $x$  may or may not enter  $B$  and if  $x$  does enter  $B$ , it only does so after  $x$  enters  $A$  (in terms of our enumeration). Since the intersection of two c.e. sets is c.e.,  $A \searrow B$  is a c.e. set. The c.e. set  $A \searrow B$  is the c.e. set of balls that first enter  $A$  and then enter  $B$  (under the above enumeration). So  $A \setminus B$  reads “ $A$  before  $B$ ” and  $A \searrow B$  reads “ $A$  before  $B$  and then  $B$ ”.

Note that for all  $W$ ,  $W \setminus A = (W - A) \sqcup (W \searrow A)$ . Since  $W \setminus A$  is a c.e. set, if  $W - A$  is not a c.e. set then  $W \searrow A$  must be infinite. (This happens for all enumerations not just our given enumeration.)

**Lemma 2.4** (Friedberg). *Assume  $A = A_0 \sqcup A_1$ , and, for all  $e$ , if  $W_e \searrow A$  is infinite then both  $W_e \searrow A_0$  and  $W_e \searrow A_1$  are infinite. Then  $A_0 \sqcup A_1$  is a Friedberg split of  $A$ .*

*Proof.* Assume that  $W - A$  is not a c.e. set but  $X = W - A_0$  is a c.e. set.  $X - A = (W - A_0) - A = W - A$  is not a c.e. set. So  $X \searrow A$  is infinite which implies that  $X \searrow A_0$  is infinite but  $X \searrow A_0 = (W - A_0) \searrow A_0 = \emptyset$ . Contradiction.  $\square$

Friedberg invented the priority method to split every c.e. set into two disjoint c.e. sets while meeting the hypothesis of the above lemma.

**Theorem 2.5** (Friedberg). *Every non-computable set  $A$  has a Friedberg split.*

*Proof.* When a ball  $x$  enters  $A$  at stage  $s$  we add it to one of  $A_0$  or  $A_1$  but which one  $x$  enters is determined by priority. Our requirements are:

$\mathcal{P}_{e,i,k}$ : if  $W_e \searrow A$  is infinite then  $|W_e \searrow A_i| \geq k$ .

We say  $x$  meets  $\mathcal{P}_{e,i,k}$  at stage  $s$  if  $|W_e \searrow A_i| = k - 1$  by stage  $s - 1$  and if we add  $x$  to  $A_i$  at stage  $s$  then  $|W_e \searrow A_i| = k$  at stage  $s$ . Find the smallest  $\langle e, i, k \rangle$  that  $x$  can meet and add  $x$  to  $A_i$  at stage  $s$ . If no such triple can be found, add  $x$  to  $A_0$  at stage  $s$ . It is not hard to show that all the  $\mathcal{P}_{e,i,k}$  are met.  $\square$

Observe that the procedure in Theorem 2.5 is uniform. Given this we made the following definition and corollary.

**Definition 2.6.** A computable function  $h$  is a *splitting procedure* iff, for all  $e$ , if  $h(e) = \langle e_0, e_1 \rangle$  then  $W_{e_0} \sqcup W_{e_1}$  is a split of  $A$  and if  $W_e$  is not computable then this split is not trivial. If  $h$  is a splitting procedure, we say that  $h(e)$  gives a split of  $W_e$  or splits  $W_e$ .

**Corollary 2.7** (of Friedberg's Proof). *There is a splitting procedure  $h$  such that if  $W_e$  is not computable then  $h(e)$  gives a Friedberg split of  $W_e$ .*

### 3. NON-TRIVIAL NON-FRIEDBERG SPLITS

The above section brings us to the following question:

**Question 3.1.** *When does a c.e. set have a non-trivial non-Friedberg split?*

This question was first asked, in a different form, as Question 1.4 in Cholak [1]. In [1], it was asked if there is a definable collection of c.e. sets such that for each set  $A$  in this collection the Friedberg splits of  $A$  are a proper subclass of the non-trivial splits of  $A$ . This question later appeared, in yet a different form, as Question 4.6 in the first unpublished version of Cholak, Gerdes, and Lange [3]. There it was suggested to compare the class of all c.e. sets all of whose non-trivial splits are Friedberg with the  $\mathcal{D}$ -maximal sets (defined below). As we will see in Theorem 3.8 every form of this question was answered by Shavrukov [11]. Shavrukov showed that a c.e. set  $A$  has a non-trivial non-Friedberg split iff  $A$  is not  $\mathcal{D}$ -maximal.

#### 3.1. There are c.e. sets with non-trivial non-Friedberg Splits.

Let  $R$  be an infinite, coinfinite, computable set. There is a non-computable c.e. subset of  $R$ , call this set  $K_R$ . There is a non-computable c.e. subset of  $\overline{R}$ , call this set  $K_{\overline{R}}$ . Let  $A = K_R \sqcup K_{\overline{R}}$ . Then  $K_R \sqcup K_{\overline{R}}$  is a non-trivial split of  $A$ .  $R - A = R - K_R$  is not c.e. but  $R - K_{\overline{R}} = R$  is a c.e. set. So this split is not Friedberg. Please note that the set  $A$  and its non-trivial non-Friedberg split are built simultaneously.

See Theorem 3.8 for more examples of sets with non-trivial non-Friedberg Splits. There are published examples of sets with non-trivial non-Friedberg splits. In Section 3.2 of Cholak, Gerdes, and Lange [3], a number of such sets are constructed. But, like in the construction in the above paragraph and Theorem 3.8, for the examples in [3] the set  $A$  and its non-trivial non-Friedberg split are built simultaneously.

#### 3.2. There are c.e. sets without non-trivial non-Friedberg Splits.

For this we need the following definitions:

**Definition 3.2.** (1)  $\mathcal{D}(A) = \{B \mid B - A \text{ is a c.e. set}\}$ .  
 (2)  $W$  is *complemented* modulo  $\mathcal{D}(A)$  iff there is a c.e.  $Y$  such that  $W \cup Y \cup A = \mathbb{N}$  and  $(W \cap Y) - A$  is a c.e. set.

- (3)  $A$  is  $\mathcal{D}$ -hhsimple iff, for every c.e.  $W$ ,  $W$  is complemented modulo  $\mathcal{D}(A)$ .
- (4) A c.e. set  $W$  is 0 modulo  $\mathcal{D}(A)$  iff  $W \in \mathcal{D}(A)$ .
- (5) A c.e. set  $W$  is 1 modulo  $\mathcal{D}(A)$  iff there is a  $Y$  such that  $Y \cap A = \emptyset$  and  $W \cup Y \cup A = \mathbb{N}$ .
- (6) A non-computable set  $A$  is  $\mathcal{D}$ -maximal iff for every  $W$ ,  $W$  is complemented modulo  $\mathcal{D}(A)$  and either 0 or 1 modulo  $\mathcal{D}(A)$ .

Assume  $W$  is 0 modulo  $\mathcal{D}(A)$ . WLOG we can assume  $W \cap A = \emptyset$ . Then  $W \cup \mathbb{N} \cup A = \mathbb{N}$  and  $\mathbb{N} \cap W = W$  is disjoint from  $A$ . So  $W$  is complemented modulo  $\mathcal{D}(A)$ . If  $W - A$  is not c.e. then  $W$  is not 0 modulo  $\mathcal{D}(A)$ . A c.e. set  $W$  is 0 modulo  $\mathcal{D}(A)$  iff  $W - A$  is a c.e. set. The set  $W$  is 1 modulo  $\mathcal{D}(A)$  as witnessed by  $Y$  iff  $W$  is complemented by  $Y$  modulo  $\mathcal{D}(A)$  and  $Y$  is 0 modulo  $\mathcal{D}(A)$ . We will not go through the details but the property of a set  $A$  being  $\mathcal{D}$ -maximal is definable in the c.e. sets,  $\mathcal{E}$ .

**Lemma 3.3** (Cholak, Downey, Herrmann). *All non-trivial splits of a  $\mathcal{D}$ -maximal set  $A$  are Friedberg.*

*Proof.* Let  $A_0 \sqcup A_1 = A$  be a non-trivial split of  $A$ . Assume that  $W - A$  is not a c.e. set. So  $W \cup A$  is 1 modulo  $\mathcal{D}(A)$ . Then, for some  $Y$ ,  $W \cup A \cup Y = \mathbb{N}$  and  $Y \cap A = \emptyset$ . If  $W - A_0$  is c.e. then  $A_0 \sqcup ((W - A_0) \cup A_1 \cup Y) = \mathbb{N}$  and hence  $A_0$  is computable. Contradiction.  $\square$

This result and the above proof explicitly appears in an earlier unpublished version of Cholak, Gerdes, and Lange [3] but not in the published version. It was first implicitly mentioned in Cholak, Downey, and Herrmann [2]. It follows a similar result about maximal sets in Downey and Stob [4].

**3.3. The Herrmann and Kummer Splitting Theorem.** Shortly we will need the following theorem.<sup>1</sup>

**Theorem 3.4** (Herrmann and Kummer Splitting Theorem). *Let  $A$  and  $B$  be c.e. sets such that  $A \subseteq B$  and  $B$  is non-complemented modulo  $\mathcal{D}(A)$ . Then there are  $B_0$  and  $B_1$  such that  $B_i$  is non-complemented modulo  $\mathcal{D}(A)$  and  $B_0 \sqcup B_1 = B$ .*

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<sup>1</sup>The Herrmann and Kummer Splitting Theorem appears, in a very different form, in Herrmann and Kummer [7]. This theorem appears in the only if direction of the proof of Theorem 2.4 of Herrmann and Kummer [7] starting on page 63 from the first full paragraph on that page. It is interesting enough to be isolated in its own right as a theorem.

*Proof.* As balls  $x$  enter  $B$  they will be enumerated into either  $B_0$  or  $B_1$ . So  $B = B_0 \sqcup B_1$ . Let  $Y_e, Z_j$  be two listings of all c.e. sets. We need to meet the requirements:

$\mathcal{R}_{e,j,i}$ : either  $B_i \cup A \cup Y_e \neq \mathbb{N}$  or  $(B_i \cap Y_e) - A \neq Z_j$ .

If we fail to meet this requirement then  $Y_e$  and  $Z_j$  witness that  $B_i$  is complemented modulo  $\mathcal{D}(A)$ .

We need a *disagreement* function. Let  $l(e, j, i, s)$  be the least  $x \leq s$  such that either  $x \notin B_{i,s} \cup A_s \cup Y_{e,s}$ , or  $x \in ((B_{i,s} \cap Y_{e,s}) - A_s)$  iff  $x \notin Z_{j,s}$ . If  $x$  does not exist, let  $l(e, j, i, s) = s$ . The  $\lim_s l(e, j, i, s)$  exists iff we will have met  $\mathcal{R}_{e,j,i}$ .

We will use  $l$  to define a *restraint* function,  $r(e, j, i, -1) = \langle e, j, i \rangle$  and  $r(e, j, i, s)$  is the max of  $r(e, j, i, s-1)$  and  $l(e, j, i, s)$ . Again, the  $\lim_s r(e, j, i, s)$  exists iff we will have met  $\mathcal{R}_{e,j,i}$ . Moreover  $r(e, j, i, s)$  is a non-decreasing function in  $s$ .

When a ball  $x$  enters  $B$  at stage  $s$  find the least  $\langle e, j, i \rangle$  such that  $x \leq r(e, j, i, s)$  and add  $x$  to  $B_i$ .

Let  $\langle e, j, i \rangle$  be the least triple such that  $\lim_s r(e, j, i, s)$  does not exist. Let  $x$  be such that for all  $\langle e', j', i' \rangle < \langle e, j, i \rangle$ ,  $\lim_s l(e', j', i', s) < x$ . Assume  $i = 0$ . Then  $B_1$  is computable (for all  $y > x$ , after  $r(e, j, 0, s) > y$ ,  $y$  cannot enter  $B_1$ ),  $Y_e$  and  $Z_j$  witness that  $B_0$  is complemented modulo  $\mathcal{D}(A)$ . Now  $Y = Y_e \cap \overline{B_1}$  and  $Z_j$  witness that  $B$  is complemented modulo  $\mathcal{D}(A)$ . Contradiction. Similarly if  $i = 1$ .  $\square$

This construction is uniform. Given an index for  $B$  we can uniformly get a split of  $B$  via the above theorem. Assume  $B$  is 0 modulo  $\mathcal{D}(A)$  witnessed by the c.e. set  $Z = B - A$ . Then  $\mathbb{N}$  and  $Z$  witness that  $B$  and any splits of  $B$  are complemented modulo  $\mathcal{D}(A)$ . Let  $e'$  and  $j'$  be the least such that  $Y_{e'} = \mathbb{N}$  and  $Z_{j'} = Z$ , and  $l(e', j', i, s) = s$  (this last item just takes playing a little with the enumeration of these sets). For some  $e \leq e', j \leq j'$  and  $i$ ,  $\lim_s r(e, j, i, s)$  does not exist and the argument above shows that the split is trivial. So if  $B \subseteq A$  this split will be trivial. So this theorem does not give rise to a splitting procedure.

If  $B$  is not complemented modulo  $\mathcal{D}(A)$  then it is open if the above split (as given above) is always Friedberg. We conjecture yes with the following evidence: We can combine the requirements  $\mathcal{P}$  from the proof of Theorem 2.5 with the one here to force the split to be a Friedberg split.

We also want to point out that the Herrmann and Kummer Splitting Theorem is very similar to the Owings Splitting Theorem.  $B$  is *non complemented modulo*  $A$  iff  $B - A$  is not co-c.e. iff  $\overline{B} \cup A$  is not c.e. The

following theorem is an easy corollary of the Owings Splitting Theorem, [10]. Also see Soare [12, X.2.5].

**Theorem 3.5** (Owings). *Let  $A$  and  $B$  be c.e. sets such that  $A \subseteq B$  and  $B$  is non-complemented modulo  $A$ . Then there are  $B_0$  and  $B_1$  such that  $B_i$  is non-complemented modulo  $A$  and  $B_0 \sqcup B_1 = B$ .*

We are not going to provide a proof. The standard proof is Soare [12, X.2.5]. What is not clear is whether this standard proof always provides a Friedberg split and, if  $B \subseteq A$ , whether the resulting split is non-trivial. We can arrange the enumeration (let  $W_0 = \mathbb{N}$ ) such that if  $B \subseteq A$  then the resulting split is non-trivial. But it is open what occurs when we use the standard enumeration. So it is unknown if the Owings Splitting Theorem gives a splitting procedure.

The Owings and the Herrmann and Kummer Splitting theorems are like Friedberg's in that all three are uniform, but unlike Friedberg's in that they do not necessarily provide non-trivial splits when possible. Herrmann and Kummer Splitting Theorem does not give rise to a splitting procedure. It is open if the Owings Splitting Theorem gives rise to a splitting procedure. Friedberg Splitting Theorem does give rise to a splitting procedure.

There is one more (little) known splitting theorem, Hammond [6], which extends all three of the splitting theorems above discussed in this subsection. Let  $\mathcal{E}$  be the collection of c.e. sets with inclusion, intersection, union,  $\emptyset$  and  $\mathbb{N}$ ; this is called the lattice of c.e. sets. An *ideal* of  $\mathcal{E}$  is a collection of sets  $\mathcal{I}$  such that  $\emptyset \in \mathcal{I}$  and  $\mathcal{I}$  is closed under subset and inclusion. An ideal  $\mathcal{I}$  is  $\Sigma_3^0$  if the relation  $W_e \in \mathcal{I}$  is  $\Sigma_3^0$ .  $\mathcal{F}$ , collection of all finite sets, is an  $\Sigma_3^0$  ideal. For any  $A$ , so are  $\mathcal{S}(A) = \{B \mid B \subseteq A\}$  and  $\mathcal{D}(A)$ .  $W$  is *complemented modulo  $\mathcal{I}$*  iff there is a  $Y$  such that  $W \cup Y = \mathbb{N}$  and  $W \cap Y$  is in  $\mathcal{I}$ . For any  $A$ , the Friedberg, Owings, and Herrmann and Kummer Splitting Theorems, respectively, imply any  $B$  which is non-complemented modulo  $\mathcal{F}$ ,  $\mathcal{S}(A)$ , or  $\mathcal{D}(A)$  can be split into  $B_0$  and  $B_1$  such that each  $B_i$  is non-complemented modulo  $\mathcal{F}$ ,  $\mathcal{S}(A)$ , or  $\mathcal{D}(A)$ .

**Theorem 3.6** (Hammond [6]). *Let  $\mathcal{I}$  be any  $\Sigma_3^0$  ideal. If  $B$  is non-complemented modulo  $\mathcal{I}$  then  $B$  can be split into  $B_0$  and  $B_1$  such that each  $B_i$  is non-complemented modulo  $\mathcal{I}$ .*

We will not include a proof here. Unlike the other three splitting theorems discussed here the proof is not finite injury. It is uniform in  $\mathcal{I}$ . Since  $\mathcal{I}$  can equal  $\mathcal{D}(A)$ , it does not always give rise to a splitting procedure. What happens when  $\mathcal{I}$  is  $\mathcal{S}(A)$  is open.



**3.4. Shavrukov's Result.** First we need to use the Herrmann and Kummer Splitting Theorem for the following corollary. The proof is not uniform.

**Corollary 3.7.** *For all non-computable non- $\mathcal{D}$ -maximal  $A$ , there are disjoint  $X_0$  and  $X_1$  such that  $X_i - A$  is not c.e. and  $A \subseteq X_0 \sqcup X_1$ .*

*Proof.* When  $A$  is not  $\mathcal{D}$ -hhsimple there is a c.e.  $X$  such that  $A \subseteq X$  and  $X$  is not complemented modulo  $\mathcal{D}(A)$ . Apply the above Herrmann and Kummer Splitting Theorem to get  $X_0 \sqcup X_1 = X$  where the  $X_i$ s are also not complemented modulo  $\mathcal{D}(A)$ . If  $X_i - A$  is c.e. then  $X_i$  is 0 and hence complemented modulo  $\mathcal{D}(A)$ . Therefore  $X_i - A$  is not a c.e. set.

Otherwise  $A$  is  $\mathcal{D}$ -hhsimple but not  $\mathcal{D}$ -maximal. So there must be a c.e. superset  $W$  of  $A$  which is not 0 or 1. So  $W - A$  is not a c.e. set. There is a  $Y$  such that  $W \cup Y = \mathbb{N}$ ,  $(W \cap Y) - A$  is c.e. but  $Y - A$  is not a c.e. set.

Let  $X_0 = W \setminus Y$  and  $X_1 = Y \setminus W$ . Now  $W = X_0 \cup (W \cap Y)$ . So  $W - A = (X_0 - A) \cup ((W \cap Y) - A)$ . The set  $(W \cap Y) - A$  is known to be c.e., so if  $X_0 - A$  is c.e. then so is  $W - A$ . Therefore  $X_0 - A$  is not a c.e. set.  $Y = X_1 \cup (W \cap Y)$ . So  $Y - A = (X_1 - A) \cup ((W \cap Y) - A)$ .  $(W \cap Y) - A$  is known to be c.e., so if  $X_1 - A$  is c.e. then so is  $Y - A$ . Therefore  $X_1 - A$  is not a c.e. set.  $\square$

**Theorem 3.8** (Shavrukov). *All c.e. non-computable non- $\mathcal{D}$ -maximal sets  $A$  have non-trivial non-Friedberg splits.*

*Proof.* By the above corollary, there are disjoint  $X_0$  and  $X_1$  such that  $X_i - A$  is not c.e. and  $A \subseteq X_0 \sqcup X_1$ . If  $X_i \cap A$  were computable then  $X_i - A = X_i \cap \overline{(X_i \cap A)}$  is c.e. Therefore  $X_0 \cap A, X_1 \cap A$  is a non-trivial split of  $A$ .  $X_0 - A$  is not c.e. but  $X_0 - (X_1 \cap A) = X_0$  is a c.e. set. Hence  $X_0 \cap A, X_1 \cap A$  is a non-trivial non-Friedberg split.  $\square$

**Corollary 3.9** (Shavrukov). *All of  $A$ 's non-trivial splits are Friedberg iff  $A$  is  $\mathcal{D}$ -maximal.*

Again we want to thank V. Yu. Shavrukov for allowing us to include his results. The proof we presented here is very different than Shavrukov's, see [11]. Shavrukov's proof used the fact that every  $\mathcal{D}$ -hhsimple is not a diagonal. For the definition of a diagonal set see Kummer [8] and Herrmann and Kummer [7].

#### 4. UNIFORM NON-TRIVIAL NON-FRIEDBERG SPLITS

The question we will answer in this section follows:



**Question 4.1.** *Is there a splitting procedure  $h$  such that all non- $\mathcal{D}$ -maximal sets  $W_e$  are split by  $h(e)$  into a non-trivial non-Friedberg split?*

The answer is no by the following theorem:

**Theorem 4.2.** *For every total computable  $h$  there is an  $e$  such that  $W_e$  is not computable and  $h(e) = \langle e_0, e_1 \rangle$  then either*

- (1)  $W_{e_0}, W_{e_1}$  is not a split of  $W_e$ ,
- (2)  $W_{e_0} \sqcup W_{e_1}$  is a trivial split of  $W_e$ , or
- (3)  $W_{e_0} \sqcup W_{e_1}$  is a Friedberg split of  $W_e$  and  $W_e$  is not  $\mathcal{D}$ -maximal.

Moreover given an index for  $h$  we can effectively find  $e$ .

Hence if  $h$  is a splitting procedure then Case (3) applies. Actually, Case (3) applies infinitely often.

**Corollary 4.3.** *Let  $h$  be a splitting procedure. Then there is an infinite set  $J$  of indices that, for all  $e \in J$ ,  $W_e$  has a non-Friedberg split but the split given by  $h(e)$  is a Friedberg split.*

*Proof of the Corollary.* Let  $h_0 = h$  and apply Theorem 4.2 to get  $e_0$ . Only Case (3) can apply. So  $W_{e_0}$  has a non-trivial non-Friedberg split but  $h(e_0)$  gives a Friedberg split. Inductively, assume for all  $j \leq i$ , that  $h_j$  and distinct  $e_j$  exist and that Case (3) applies to  $W_{e_j}$ . Let  $W_{a_i} \sqcup W_{b_i}$  be a non-trivial non-Friedberg split of  $W_{e_i}$ . Let  $h_{i+1}(e_i) = \langle a_i, b_i \rangle$  and if  $e \neq e_i$  let  $h_{i+1}(e) = h_i(e)$ . Apply Theorem 4.2 to  $h_{i+1}$  to effectively get an  $e_{i+1}$ . Case (3) applies to  $e_{i+1}$  and  $e_{i+1} \neq e_j$ , for all  $j \leq i$ . Let  $J$  be the infinite set  $\{e_i | i \in \omega\}$ .  $\square$

We can create a splitting procedure that is correct on infinite many indices of a non- $\mathcal{D}$ -maximal set. Take  $A_0 \sqcup A_1 = A = W_a \sqcup W_b = W_c$  to be a non-trivial non-Friedberg splitting of  $A$ . Using the padding lemma, let  $I$  be an infinite computable set of indices for  $A$ . Define  $h(e)$  to be  $\langle a, b \rangle$  if  $e \in I$  and  $h_F(e)$  otherwise, where  $h_F$  is from Corollary 2.7. By Rice's Theorem,  $I$  is not all indices for  $A$ . But the following is open.

**Question 4.4.** *Is there a splitting procedure  $h$  and a c.e. set  $A$  with a non-trivial non-Friedberg split such that, if  $W_e = A$  then  $h(e)$  gives a non-trivial non-Friedberg split of  $W_e = A$ ?*

## 5. PROOF OF THEOREM 4.2

The goal of the rest of the paper is to provide a proof of the above Theorem 4.2. Assume that we are given  $h$  and we will construct  $A$ . Via the Recursion Theorem we can assume that  $W_e = A$ . Also assume that  $h(e) = \langle e_0, e_1 \rangle$ .

For our proof we will work using an oracle for certain  $\Pi_2^0$  questions. Certainly  $\mathbf{0}''$  works but is overkill. The index set of all infinite c.e. sets works nicely. We will use a tree argument to provide answers to our  $\Pi_2^0$  questions. The tree will also provide a framework for our construction.

We will build  $A$  in pieces. First we will construct a  $\Delta_3^0$  list of pairwise disjoint computable sets  $R$  such that every c.e. set or its complement will be in the union of finitely many of these computable sets and the union of all them is  $\mathbb{N}$ . Inside each of these computable sets we will build a piece of  $A$ . The default is that  $A$  will be maximal inside each  $R$  but finite or cofinite inside  $R$  are also possible. The construction will ensure that the union of these pieces is a c.e. set  $A$ . If  $A$  is maximal in only finitely many of these computable sets then  $A$  will turn out to be  $\mathcal{D}$ -maximal.

We will *try* to construct infinite, coinfinite, computable sets  $R_i$  such that, for all  $j$ , either

$$(5.0.1) \quad W_j \subseteq^* \bigcup_{i \leq j} R_i \cup A,$$

or

$$(5.0.2) \quad W_j \cup \bigcup_{i \leq j} R_i \cup A =^* \mathbb{N}.$$

(We will remind the reader that  $X =^* Y$  iff  $(X - Y) \sqcup (Y - X)$  is finite.) Since these sets are meant to be computable we also have to build  $\overline{R}_i$  while we are building  $R_i$ . Assume that we have built the sets  $R_i$  up to  $j$ . The balls in  $\bigcap_{i < j} \overline{R}_i$  have not yet been added to  $R_j$  or  $A$ . So our construction will ensure  $(\bigcap_{i < j} \overline{R}_i) \setminus A$  is infinite. To build  $R_j$  ask if

$$(5.0.3) \quad P_j = (W_j \cap \bigcap_{i < j} \overline{R}_i) \setminus A$$

is infinite. This is a  $\Pi_2^0$  question. If  $P_j$  is infinite, we will build  $\overline{R}_j$  as a subset of  $W_j$ , so that Equation 5.0.2 is satisfied. When we add balls from the set  $\bigcap_{i < j} \overline{R}_i$  to  $R_j$ , we will make sure that there is at least one ball in  $W_j \cap \bigcap_{i < j} \overline{R}_i$  currently uncommitted. We will add that ball to  $\overline{R}_j$  and the rest of the balls under consideration to  $R_j$ . We will do this infinitely often. In this case, we satisfy Equation 5.0.2. If  $P_j$  is finite, then, since  $(\bigcap_{i < j} \overline{R}_i) \setminus A$  is infinite, we just build  $R_j$  and  $\overline{R}_j$  to be infinite within  $\bigcap_{i < j} \overline{R}_i$  and Equation 5.0.1 is satisfied.

Now inside each  $R_i$  we will build  $A$  to be finite, cofinite, or maximal depending on various outcomes. The default will be for  $A$  to be maximal in  $R_i$ . To do this we use the construction presented in Soare [12, X.3.3] as a guide to work inside  $R_i$ . We will go over the details later. Since maximal sets are not computable,  $A$  will not be computable. Assume that  $A$  is maximal inside  $R_i$  and  $R_l$ , where  $l \neq i$ , then, since  $A \cap R_l$  is a non-computable subset of  $A \cap \overline{R_i}$ ,  $A = (A \cap R_i) \sqcup (A \cap \overline{R_i})$  is a non-trivial non-Friedberg split of  $A$ . The details follow the construction in Subsection 3.1. Now by Theorem 3.8,  $A$  is not  $\mathcal{D}$ -maximal. If  $W_{e_0} \sqcup W_{e_1}$  is not a split of  $A$  then we are done. So we may as well assume that  $W_{e_0} \sqcup W_{e_1}$  is a split of  $A$ .

We will now consider how this split behaves inside each  $R_i$ . Since  $A$  is maximal inside  $R_i$  there are two choices either the split is trivial or Friedberg. We are going to ask an infinite series of questions designed to tell if the split inside  $R_i$  is trivial. The questions are is “ $W_k \sqcup (W_{e_0} \cap R_i) = R_i$ ” and is “ $W_k \sqcup (W_{e_1} \cap R_i) = R_i$ ”, for all  $k$ . Again these questions are  $\Pi_2^0$ . A positive answer will tell us the split is trivial inside  $R_i$  and which set  $W_{e_0} \cap R_i$  or  $W_{e_1} \cap R_i$  is computable.

Assume that we get a positive answer and the information that the set  $W_{e_0} \cap R_i$  is computable. In this case we will take the following action: Dump almost all of  $R_l$ , for  $l < i$ , into  $A$  and, for  $l > i$ , stop adding balls from  $R_l$  into  $A$ . In fact, stop building  $R_l$ . In this case,  $A$  is computable outside  $R_i$  and hence  $W_{e_0}$  must also be computable. So  $W_{e_0} \sqcup W_{e_1}$  is a trivial split of  $A$ . We act similarly if  $W_{e_1} \cap R_i$  is computable.

If none of the answers to these questions for each  $R_i$  is positive then  $W_{e_0} \sqcup W_{e_1}$  is a non-trivial split of  $A$ . We know inside each  $R_i$  the split is Friedberg. We must show that globally the split is Friedberg. Let's consider  $W_j$ . If Equation 5.0.1 holds, then  $W_j - A \subseteq^* \bigsqcup_{i \leq j} R_i$ . So, if  $W_j - A$  is not a c.e. set neither are  $W_j - W_{e_0}$  and  $W_j - W_{e_1}$ . So assume Equation 5.0.2 holds,  $W_j - A$  is not a c.e. set, but  $W_j - W_{e_0}$  is a c.e. set. For any  $n > j$ ,  $(W_j - A) \cap R_n$  cannot be a c.e. set. But  $(W_j - W_{e_0}) \cap R_n$  is a c.e. set. This contradicts that our split is Friedberg inside  $R_n$ . A similar argument works if Equation 5.0.2 holds,  $W_j - A$  is not a c.e. set, but  $W_j - W_{e_1}$  is a c.e. set. Our split is a Friedberg split.

With one positive answer, we must take action to ensure that our given split is trivial. One positive answer is a  $\Sigma_3^0$  event. If all questions have negative answers then we have a  $\Pi_3^0$  event and, in this case, our split is a Friedberg split.

**5.1. Coding our  $\Pi_2^0$  Questions via a Tree.** We will work with the tree,  $2^{<\omega}$ . We consider the tree to grow downward. At the empty node,

$\lambda$ , we will construct  $A$  and  $\overline{R}_\lambda = \tilde{R}_\lambda = \mathbb{N}$ . At nodes  $\alpha$  of length  $i^2 > 0$  we will construct  $R_\alpha$  and  $\tilde{R}_\alpha$  ( $\overline{R}_\alpha = \sqcup_{\beta \subset \alpha} R_\beta \sqcup \tilde{R}_\alpha$ .) We will call such nodes  $R$ -nodes. The idea is that if  $f$  is the true path,  $|\alpha| = i^2$ , and  $\alpha \prec f$ , then  $R_\alpha = R_i$  and  $\sqcup_{\beta \subset \alpha} R_\beta \sqcup \tilde{R}_\alpha = {}^* \mathbb{N}$ . (We will start indexing the  $R_i$  at 1.)

Since we need to ask questions about the potential  $R_i$ 's we need the indices for the  $R_\alpha$ 's. So the real outcome of our construction is a pair of functions  $g$  and  $\tilde{g}$  such that  $W_{g(\lambda)} = A$ ,  $W_{g(\alpha)} = R_\alpha$ , and  $W_{\tilde{g}(\alpha)} = \tilde{R}_\alpha$ , for all  $\alpha$ . Via the Recursion Theorem, we can assume we know  $g$  and  $\tilde{g}$  prior to the construction. We will use this knowledge to code our questions into the tree.

Let  $|\gamma| = j^2 - 1$ . Let  $\delta \subset \gamma$  such that  $|\delta| = (j-1)^2$ . At  $\gamma$  we will code the question “Is  $(W_j \cap \tilde{R}_\delta) \setminus A$  infinite?”. *Strictly* between two  $R$ -nodes of length  $j^2$  and  $(j+1)^2$  there are  $((j+1)^2 - 1) - j^2 = 2j$  nodes. If  $|\gamma| = j^2 + 2k - 2$ ,  $1 \leq k \leq j$ ,  $\beta \preceq \gamma$ , and  $|\beta| = k^2$ , then at  $\gamma$  code the question “Does  $W_j \sqcup (W_{e_0} \cap R_\beta) = R_\beta$ ?”. If  $|\gamma| = j^2 + 2k - 1$ ,  $1 \leq k \leq j$ ,  $\beta \preceq \gamma$ , and  $|\beta| = k^2$ , then at  $\gamma$  code the question “Does  $W_j \sqcup (W_{e_1} \cap R_\beta) = R_\beta$ ?”. (The only difference in these two sentences is the length of  $\gamma$  differences by 1 and the second uses  $W_{e_1}$  rather than  $W_{e_0}$ .)

Via the use of the Recursion Theorem, as we discussed two paragraphs above, these are uniformly  $\Pi_2^0$  questions. There is a uniform reduction from these questions to the index set of infinite c.e. sets or INF. So uniformly, for all  $\gamma$ , we can associate a c.e. *chip* set  $C_\gamma$  such that  $C_\gamma$  is infinite iff the question coded at  $\gamma$  has a positive answer.

Earlier we have called some nodes  $R$ -nodes. These were the nodes whose length is a perfect square. Other than the empty node, we will call the remaining nodes  $A$ -nodes; they provide answers to questions coded at  $\alpha$ 's predecessor,  $\alpha^- = \gamma$ . We call an  $A$ -node  $\alpha$  *positive* iff  $\alpha \hat{=} 1 = \gamma$ . Otherwise an  $A$ -node is negative.

We will inductively define the *true path*,  $f$ .  $\lambda$  is on  $f$ . Assume that  $\alpha \preceq f$ . If  $\alpha$  is a positive  $A$ -node then  $f = \alpha$ . Otherwise,  $\alpha \hat{=} 1 \prec f$  iff  $C_\alpha$  is infinite and  $\alpha \hat{=} 0 \prec f$  iff  $C_\alpha$  is finite. Since nodes of length 0 and 1 are not  $A$ -nodes, there is always an  $R$ -node on the true path. Either all the  $A$ -nodes on  $f$  are negative or  $f$  is finite and ends with a positive  $A$ -node.

A key to the construction is the approximation to the true path at stage  $s$ ,  $f_s$ . Define  $f_0 = \lambda$ , the empty node. Assume that  $\alpha \subseteq f_{s+1}$  and  $|\alpha| < s^2$ . If  $\alpha$  is a positive  $A$ -node, let  $f_{s+1} = \alpha$ . Assume that  $\alpha$  is not a positive  $A$ -node. Let  $t$  be the greatest stage less than  $s+1$  such

that  $\alpha \subseteq f_t$ . If no such stage exists, let  $t = 0$ . If  $C_{\alpha,t} \neq C_{\alpha,s+1}$  then let  $\alpha \hat{1} \subseteq f_{s+1}$ . Otherwise,  $\alpha \hat{0} \subseteq f_{s+1}$ .

Since nodes of length  $s^2$  are  $R$ -nodes, for  $s > 0$ ,  $f_s$  always ends in an  $R$ -node or a positive  $A$ -node. We say  $\alpha <_L \beta$  (or  $\alpha$  is to the left of  $\beta$ ) iff  $\alpha \subsetneq \beta$  or there is a  $\gamma$  such that  $\gamma \hat{1} \subseteq \alpha$  and  $\gamma \hat{0} \subseteq \beta$ . By induction on  $l$ , we can show that  $\liminf_s f_s \upharpoonright l = f \upharpoonright l$  (the  $\liminf$  is measured w.r.t.  $<_L$ ). So,  $\liminf_s f_s = f$ . If  $f_s <_L \alpha$  then there is always a least (in terms of length)  $R$ -node or positive  $A$ -node,  $\beta$ , such that  $\beta \subseteq f_s$  and  $\beta <_L \alpha$ .

**5.2. Action on the Tree.** We will use the tree and  $f_s$  to construct  $A$ ,  $R_\alpha$ , and  $\tilde{R}_\alpha$ , for all  $\alpha$ . We think of this construction as a pinball machine. Integers or balls enter at top node,  $\lambda$ , and move downwards and leftwards. The position of a ball,  $x$ , at the end of stage  $s$  is given by the function  $\alpha(x, s)$ . The movement on the tree is done such that the  $\lim_s \alpha(x, s)$  exists. Let  $\alpha(x) = \lim_s \alpha(x, s)$ . Initially,  $\alpha(x, s)$  is not defined (so  $x$  is not on the machine) and, unless explicitly changed,  $\alpha(x, s)$  remains the same from stage to stage. For the balls on the machine, at every stage  $s$ ,  $\alpha(x, s)$  is always an  $R$ -node or a positive  $A$ -node, and  $|\alpha(x, s)| \leq x^2$ . (The bound  $x^2$  was chosen since balls can only rest at  $R$ -nodes or positive  $A$  nodes and the length of  $R$ -nodes are perfect squares.) If a ball  $x$  enters  $A$  at stage  $s$ ,  $x$  is removed from the tree at stage  $s$  and  $\alpha(x, s)$  is undefined again.

Entering the machine and leftward movement is determined by  $f_{s+1}$ . Downward movement will be discussed later. Let  $\beta$  be the  $R$ -node of length 1 such that  $\beta \subseteq f_{s+1}$ . Let  $\alpha(s, s+1) = \beta$ . So all the balls on the machine at stage  $s$  are less than  $s$ . Assume that  $\alpha(x, s) = \alpha$  and  $f_{s+1} <_L \alpha$ . Then there is always a least (in terms of length)  $R$ -node or positive  $A$ -node,  $\beta$ , such that  $\beta \subseteq f_{s+1}$ ,  $\beta \not\subseteq \alpha$ , and  $\beta <_L \alpha$ . Let  $\alpha(x, s+1) = \beta$ . Since  $|\alpha| \leq x^2 + 1$ , the same is true for  $\beta$ . A ball  $x$  can only move leftward finitely many times. Since  $\liminf_s f_s = f$ ,  $\alpha(x) <_L f$  or  $\alpha(x) \subseteq f$ .

Assume that  $\alpha$  is an  $R$ -node. So the length of  $\alpha$  is  $j^2$  for some  $j$ . Either  $\alpha = \alpha^- \hat{1}$  or  $\alpha = \alpha^- \hat{0}$ . At  $\gamma = \alpha^-$  we asked the question “Is  $(W_j \cap \tilde{R}_\delta) \setminus A$  infinite?”, where  $\delta$  is the greatest proper  $R$ -subnode of  $\gamma$ . If  $\alpha$  ends with a 1, then  $\alpha$  believes this set is infinite. If  $\alpha$  ends with a 0 then  $\alpha$  believes this set is finite. If  $\alpha$  ends with a 1 let  $P_\alpha = (W_j \cap \tilde{R}_\delta) \setminus A$ . Otherwise, let  $P_\alpha = \tilde{R}_\delta \setminus A$ . We also defined  $P_\alpha$  for positive  $A$ -nodes to be  $P_\alpha = \tilde{R}_\delta \setminus A$ , where  $\delta \subset \alpha$  is the greatest  $R$ -node contained in  $\alpha$ .  $\alpha$  wants all balls in  $P_\alpha$  to go through  $\alpha$ . Moreover the construction of  $A$  inside  $R_\alpha$  requires that  $\alpha$  see fresh balls in  $P_\alpha$ . So these  $\alpha$  are allowed to pull balls in  $P_\alpha$ .

We will now work on the remaining movement, pulling, on our pinball machine. An  $R$ -node or positive  $A$ -node  $\alpha$  is allowed to pull balls from subnodes of  $\alpha$  or nodes to the right of  $\alpha$ . Pulling will be downward or leftward movement. The only downward movement allowed is done via pulling. When  $\alpha$  can pull balls is controlled by  $f_s$ . When  $\alpha \subseteq f_s$ ,  $\alpha$  puts a request coded by  $s$  on a list denoted by  $\mathcal{P}_\alpha$  at stage  $s$ .  $\alpha$  can only pull balls when there is an unfulfilled request on the list. If  $\alpha$  takes action (as described below) at stage  $s$  then the least request on  $\mathcal{P}_\alpha$  has been fulfilled. If  $f_s <_L \alpha$  then all the current requests at stage  $s$  on  $\mathcal{P}_\alpha$  are declared fulfilled.

Let  $\alpha$  be an  $R$ -node or  $A$ -node of length  $l$  and assume that there is an unfulfilled request on  $\mathcal{P}_\alpha$  at stage  $s$ . Assume that there are two different balls,  $x_0$  and  $x_1$ , such that  $x_i > l$ ,  $x_i \in P_{\alpha,s}$ , and either  $\alpha <_L \alpha(x_i, s)$  ( $x_i$  is to the right of  $\alpha$ ) or  $\alpha(x_i, s) \subset \alpha$  ( $x_i$  is above  $\alpha$ ). For leftmost  $\alpha$  and the least such pair, at stage  $s + 1$ , take the following action: Let  $\alpha(x_i, s + 1) = \alpha$  and, if  $\alpha$  is a  $R$ -node, then put  $x_0$  into  $R_{\alpha,s+1}$  and put  $x_1$  into  $\tilde{R}_{\alpha,s+1}$ . For all balls  $y$ , such that  $|\alpha|^2 < y < \max x_i$ ,  $y \in \tilde{R}_{\delta,s}$  (using the above notation for  $\delta$ ), and either  $\alpha <_L \alpha(x_i, s)$  or  $\alpha(x_i, s) \subset \alpha$ , let  $\alpha(y, s + 1) = \alpha$  and, if  $\alpha$  is  $R$ -node, then add  $y$  to  $R_{\alpha,s+1}$ . This request is declared fulfilled.

There is just a little more to the construction of  $R_\alpha$ . In the next section we will discuss the construction of  $A$  inside  $R_\alpha \setminus A$ . Recall earlier that we said that if a ball enters  $A$  it is removed from the machine. That means that none of the above balls added to  $R_\alpha$  and  $\tilde{R}_\alpha$  are in  $A$ . To make sure that  $R_\alpha$  is computable when  $\alpha \subset f$  we must be sure that almost all balls from  $\tilde{R}_\delta$  enter  $R_\alpha$  or  $\tilde{R}_\alpha$ . Because of the construction to the right of the true path, infinitely many balls in  $\tilde{R}_\delta$  might enter  $A$  before they enter  $R_\alpha$  or  $\tilde{R}_\alpha$ . The balls we are talking about are in the c.e. set  $(\tilde{R}_\delta \searrow A) \setminus (R_\alpha \sqcup \tilde{R}_\alpha)$ . The above action cannot add these balls to  $R_\alpha$  or  $\tilde{R}_\alpha$ . So we will simply add these balls to  $R_\alpha$ . So the above set is equal to  $A \searrow R_\alpha$ .

Let's see inductively that for  $\alpha \subset f$  and  $\alpha$  is an  $R$ -node, that  $R_\alpha \setminus A$  is infinite,  $\tilde{R}_\alpha \setminus A$  is infinite,  $\bigsqcup_{\beta \subseteq \alpha} R_\beta \sqcup \tilde{R}_\alpha =^* \mathbb{N}$ , and  $A \searrow R_\alpha$  is computable. Let  $\delta$  be the greatest proper  $R$ -subnode of  $\alpha$ . If no such node exists let  $\delta = \lambda$ . So by our inductive hypothesis  $\tilde{R}_\delta \setminus A$  is infinite. Moreover, by the movement on the tree, only finite many of these balls are ever to the left of  $\alpha$ . Ignore those balls. Since  $\alpha$  is on the true path, infinitely many requests are placed on  $\mathcal{P}_\alpha$  and only finitely many of them are fulfilled because  $f_s <_L \alpha$ . We claim all of the remaining requests are fulfilled. If not then all but finitely balls of  $P_\alpha$  can be pulled by  $\alpha$  and  $\alpha$  will eventually pull two balls fulfilling the desired

request. So the action discussed two paragraphs above occurs infinitely often. We have ensured that  $R_\alpha \setminus A$  and  $\tilde{R}_\alpha \setminus A$  are infinite. The sets  $R_\beta$ , for  $\beta \subseteq \alpha$ , and  $\tilde{R}_\alpha$  are all pairwise disjoint. By the action in the above paragraph the union of all these sets is almost everything. Since the disjoint union of these sets is almost everything, we also have that  $A \searrow R_\alpha$  is computable. Moreover if  $\alpha$  ends with a 1 then  $\tilde{R}_\alpha \subseteq W_j$ , where  $|\alpha| = j^2$ , and hence  $W_j \cup A \cup \bigcup_{\beta \subseteq \alpha} R_\beta = {}^*\mathbb{N}$  and Equation 5.0.2 holds. If  $\alpha$  ends with a 0 then  $W_j \subseteq {}^*A \cup \bigcup_{\beta \subseteq \alpha} R_\beta$  and Equation 5.0.1 holds.

Assume  $f$  is finite. So  $\alpha = f$  is a positive  $A$  node. Let  $\gamma$  be the greatest  $R$ -subnode of  $\alpha$ . Let  $Z$  be the set of  $x$  such that there is a stage  $s$  where  $\alpha(x, s) = \alpha$ .  $Z$  is a c.e. set. Because  $\alpha = f$  for almost all balls  $x$  in  $Z$ ,  $\alpha(x) = \alpha$ . Almost all of the balls in  $Z$  never enter  $A$ .  $Z$  is the end of the line. Recall that  $P_\alpha = \tilde{R}_\delta \setminus A$ . By the pulling action almost all of the balls in  $P_\alpha$  will enter  $Z$ . By the above paragraph,  $\bigcup_{\beta \subseteq \delta} R_\beta \sqcup \tilde{R}_\delta = {}^*\mathbb{N}$ . So  $Z$  and  $A \searrow \tilde{R}_\delta$  are computable sets.

**5.3. The construction of  $A$ .** We will build  $A$  to be maximal inside  $R_\alpha \setminus A$ . Since  $\alpha \subset f$ ,  $R_\alpha \setminus A$  is an infinite computable set. Let  $R = R_\alpha \setminus A$ . We build  $A \cap R$  stagewise based on the construction of a maximal set from Soare [12, Theorem X.3.3].

The main requirement is to ensure that, for all  $e$ ,

$$\mathcal{M}_e : \quad W_e \cap R \subseteq {}^*A \cap R \text{ or } (W_e \cap R) \cup (A \cap R) = R.$$

$\sigma(e, x, s) = \{i : i \leq e \wedge x \in W_{i,s}\}$  is the  $e$ -state of  $x$  at stage  $s$ . We will have a series of markers  $\Gamma_n^\alpha$  with  $a_n^{\alpha,s}$  denoting the position of  $\Gamma_n^\alpha$  at stage  $s$  and such that  $\overline{A}_s \cap R = \{a_0^{\alpha,s} < a_1^{\alpha,s} \dots\}$ . Each marker  $\Gamma_e$  wants to move to maximize the  $e$ -state of  $a_e^\alpha = \lim a_e^{\alpha,s}$ .

Initially, we let  $A_0 \cap R = \emptyset$  and define the  $a_n^{\alpha,0}$  accordingly. At stage  $s+1$ , if there is a least  $e$  such that for some least  $i$ ,  $e < i < s$  and  $\sigma(e, a_i^{\alpha,s}, s) > \sigma(e, a_e^{\alpha,s}, s)$ , then we dump  $a_e^{\alpha,s}, a_{e+1}^{\alpha,s}, \dots, a_{i-1}^{\alpha,s}$  into  $A$  at stage  $s+1$ . So  $a_e^{\alpha,s+1} = a_i^{\alpha,s}$ . Let's call this dumping the *original dumping*. If  $e$  does not exist do nothing.

Certain positive  $A$ -nodes  $\gamma$  below  $\alpha$  can also dump balls from  $R = R_\alpha \setminus A$  into  $A$ . Let  $\gamma$  be a positive  $A$ -node such that  $\alpha \subset \gamma$  and at  $\gamma^-$  is coded the question “Is  $W_j \sqcup (W_{e_0} \cap R_\beta) = R_\beta$  infinite?” or “Is  $W_j \sqcup (W_{e_1} \cap R_\beta) = R_\beta$  infinite?”, for some  $j$  and some  $\beta \neq \alpha$ .  $\gamma$  believes that our split is trivial inside some  $R_\beta$  and wants to dump almost all of  $R_\alpha$  into  $A$ . Let  $t_{\gamma,s}$  be the maximum of  $|\gamma|$  and the greatest stage  $t$  such that  $t \leq s$  and  $f_t <_L \gamma$ . Assume  $\gamma \subset f_{s+1}$ , dump  $a_s^{\alpha,t_{\gamma,s}}$  into  $A$  at stage  $s+1$  (if the above movement of balls at stage  $s+1$  has already



forced  $a_s^{\alpha, t_{\gamma, s}} \neq a_{s+1}^{\alpha, t_{\gamma, s}}$  that is enough). Let's call this dumping, *extra dumping*.

The positive  $A$ -nodes to the left or to the right of the true path only dump  $a_s^{\alpha, e}$  finitely often. The ones to the left of the true path are only on  $f_s$  finitely often and hence only dump finitely many balls from  $R_\alpha \setminus A$  into  $A$ . If  $f <_L \gamma$  then  $\lim_s t_{\gamma, s}$  goes to infinity and  $\gamma$  can only dump each  $a_s^{\alpha, e}$  into  $A$  finitely often.

Assume  $\gamma = f_s$  is a positive  $A$ -node and  $\alpha \subset \gamma$  and at  $\gamma^-$  is coded the question “Is  $W_j \sqcup (W_{e_0} \cap R_\beta) = R_\beta$  infinite?” or “Is  $W_j \sqcup (W_{e_1} \cap R_\beta) = R_\beta$  infinite?”, for some  $j$  and some  $\beta \neq \alpha$ . Then  $\lim_s t_{\gamma, s}$  exists and almost all balls in  $R_\alpha$  are dumped into  $A$ , i.e.  $(R_\alpha \setminus A) \subseteq^* A$ . For the rest of this section we will assume the above assumption is false.

So the extra dumping at most dumps each  $a_s^{\alpha, e}$  into  $A$  finitely often. Assume that  $a_s^{\alpha, e}$  will not be dumped after stage  $s$  via our extra dumping. Since the original dumping only dumps to increase the  $e$ -state and there are  $2^e$  many  $e$ -states, the original dumping only dumps  $a_s^{\alpha, e}$  finitely often. Hence  $\lim_s a_s^{\alpha, e}$  exists and equals  $a^{\alpha, e}$ .

Now we are in a position to show that the requirements  $\mathcal{M}_e$  are met. Assume that  $\mathcal{M}_i$  holds for  $i < e$  and there is an  $(e - 1)$ -state  $\tau$  such that almost all of  $R - A$  are in state  $\tau$ . Assume all balls greater than  $k$  in  $R - A$  are in state  $\tau$ . Let

$$M = \{x : \exists s, n [\sigma(e - 1, x, s) = \tau \wedge n \geq k \wedge x = a_s^{\alpha, n}]\}.$$

So  $R - A \subseteq^* M$ . Assume  $(M \cap W_e) \setminus A$  is finite. Then  $W_e \cap R \subseteq^* A \cap R$  and almost all balls in  $R - A$  are in  $e$ -state  $\tau$ . Now assume  $(M \cap W_e) \setminus A$  is infinite. Let  $n \geq k$  and  $\sigma(e, a_s^{\alpha, n}, s) = \tau$ . Since eventually there will be an  $m$  and stage  $t$  where  $\sigma(e, a_t^{\alpha, m}, t) = \tau \cup \{e\}$ ,  $a^{\alpha, n} \neq a_s^{\alpha, n}$ . So  $R - A \subseteq^* W_e$ . So,  $A$  is maximal inside  $R_\alpha$ .

**5.4. Putting it all together.** Recall that if  $\alpha = f$  is a positive  $A$ -node then, for some  $j$ , some  $i$ , and some  $\beta \subset f$ ,  $W_j$  witnesses that  $W_{e_i} \cap A$  is a computable subset of  $R_\beta$ . In this case, a  $\Sigma_3^0$  event occurs. By the work in the above paragraph, we know that  $A \cap R_\beta$  is maximal in  $R_\beta$  and hence  $A \cap R_\beta$  is not computable. So  $A$  is not computable. So if  $W_{e_0} \sqcup W_{e_1}$  is not a split of  $A$  we are done. Assume otherwise. By our assumption, inside  $R_\beta$ ,  $W_{e_0} \sqcup W_{e_1}$  is the trivial split. Let  $\delta$  be the greatest  $R$ -subnode of  $\alpha$ . By work in the last paragraph of Section 5.2, there is a set  $Z$ , such that  $\bigsqcup_{\gamma \subseteq \delta} R_\gamma \sqcup (A \setminus \bar{R}_\delta) \sqcup Z =^* \mathbb{N}$  and  $Z \setminus A = \emptyset$ . Now, by the above section, for all  $\gamma$ , such that  $\gamma \subset \alpha$  and  $\gamma \neq \beta$ ,  $A \cap R_\gamma =^* R_\gamma$ . Therefore outside of  $R_\beta$ ,  $A$  is computable; i.e.  $A \cap \bar{R}_\beta$  is computable. Any split of a computable set is trivial.

Therefore,  $W_{e_0} \sqcup W_{e_1}$  is a trivial split of  $A$ . So, by Theorem 4.2,  $A$  is  $\mathcal{D}$ -maximal.

For the remaining part of this paper, assume that  $f$  is an infinite path through  $2^{<\omega}$ . So a  $\Pi_3^0$  event occurs. In this case, by the above section, for all  $\alpha \subset f$ , where  $\alpha$  is an  $R$ -node,  $A \cap R_\alpha$  is not computable. So  $A$  is not computable. If  $W_{e_0} \sqcup W_{e_1}$  is not a split of  $A$  we are done. Assume otherwise. Let  $\alpha$  be any  $R$ -node where  $\alpha \subset f$ . Let  $A = (A \cap R_\alpha) \sqcup (A \cap \overline{R_\alpha})$ . There is an  $R$ -node  $\beta \neq \alpha$  on the true path.  $A \cap R_\beta$  is also not computable. Hence,  $(A \cap \overline{R_\alpha})$  is not computable and  $A = (A \cap R_\alpha) \sqcup (A \cap \overline{R_\alpha})$  is a non-trivial split of  $A$ . The split is not Friedberg, since  $R_\alpha - A$  is not a c.e. set but  $R_\alpha - (A \cap \overline{R_\alpha}) = R_\alpha$  is computable. Therefore, by Theorem 4.2,  $A$  is not  $\mathcal{D}$ -maximal.

It just remains to show that  $W_{e_0} \sqcup W_{e_1}$  is a Friedberg split of  $A$ . We know that for all  $\gamma \subset f$ ,  $W_{e_0} \sqcup W_{e_1}$  is a Friedberg split of  $A$  inside  $R_\gamma$ . Since splits of maximal sets are either trivial or Friedberg, otherwise the above  $\Sigma_3^0$  event occurs. We must show that globally the split is Friedberg. Let's consider  $W_j$ . Let  $\alpha \subset f$  such that  $|\alpha| = j^2$ . By the work in the second to last paragraph of 5.2, either  $W_j \subseteq^* A \cup \bigsqcup_{\beta \subseteq \alpha} R_\beta$  or  $W_j \cup A \cup \bigsqcup_{\beta \subseteq \alpha} R_\beta =^* \mathbb{N}$ . In the first case, if  $W_j - A$  is not a c.e. set neither are  $W_j - W_{e_0}$  and  $W_j - W_{e_1}$ . Assume that  $W_j \cup A \cup \bigsqcup_{\beta \subseteq \alpha} R_\beta =^* \mathbb{N}$ . Furthermore, assume  $W_j - A$  is not a c.e. set, but  $W_j - W_{e_0}$  is a c.e. set. For any  $\gamma$ , where  $\alpha \subset \gamma \subset f$ ,  $(W_j - A) \cap R_\gamma$  cannot be a c.e. set since this set contains  $R_\gamma - A$  and  $A$  is maximal inside  $R_\gamma$ . But, since  $W_j - W_{e_0}$  is a c.e. set,  $(W_j - W_{e_0}) \cap R_\gamma$  is a c.e. set. This contradicts the fact that our split is Friedberg inside  $R_\gamma$ . A similar argument works if  $W_j - A$  is not a c.e. set, but  $W_j - W_{e_1}$  is a c.e. set. So our split is a Friedberg split.  $\square$

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